

POINCARÉ DUALITY FOR SPACES WITH ISOLATED SINGULARITIES

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ABSTRACT. In this paper we assign, under reasonable hypothesis, to each pseudomanifold with isolated singularities a rational Poincaré duality space. These spaces are constructed with the formalism of intersection spaces defined by Markus Banagl and are indeed related to them in the even dimensional case.

1. INTRODUCTION

We are concerned with rational Poincaré duality for singular spaces. There is at least two ways to restore it in this context :

- As a self-dual sheaf. This is for instance the case with rational intersection homology.
- As a spatialization. That is, given a singular space X , trying to associate to it a new topological space X_{DP} that satisfies Poincaré duality. This strategy is at the origin of the concept of intersection spaces.

Let us briefly recall this two approaches.

While seeking for a theory of characteristic numbers for complex analytic varieties and other singular spaces, Mark Goresky and Robert MacPherson discovered (and then defined in [9] for PL pseudomanifolds and in [8] for topological pseudomanifolds) a family of groups $IH_*^{\bar{p}}(X)$ called intersection homology groups of X . These groups depend on a multi-index \bar{p} called a perversity. Intersection homology is able to restore Poincaré duality on topological stratified pseudomanifolds.

If X is a compact oriented pseudomanifold of dimension n and \bar{p}, \bar{q} are two complementary perversities, over \mathbf{Q} we have an isomorphism

$$IH_r^{\bar{p}}(X) \cong IH_q^{n-r}(X),$$

With $IH_q^{n-r}(X) := \text{hom}(IH_{n-r}^{\bar{q}}(X), \mathbf{Q})$.

Intersection spaces were defined by Markus Banagl in [2] as an attempt to spatialize Poincaré duality for singular spaces. Given a stratified pseudomanifold X of dimension n with only isolated singularities and simply connected links we have a family of topological spaces $I^{\bar{p}}X$ indexed by perversities \bar{p} . By analogy with intersection homology, denote by $\tilde{H}I_*^{\bar{p}}(X) := \tilde{H}_*(I^{\bar{p}}X)$ and $\tilde{H}I_{\bar{p}}^*(X) := \tilde{H}^*(I^{\bar{p}}X)$. Over \mathbf{Q} and for complementary perversities \bar{p}, \bar{q} , we have an isomorphism

$$\tilde{H}I_r^{\bar{p}}(X) \cong \tilde{H}I_q^{n-r}(X).$$

One may regard the theory of intersection spaces as an enrichment of intersection homology and we recover informations about intersection homology thanks to those intersection spaces. In particular they have the same information when it comes to the signature of the intersection form as shown in [2, theorem 2.28]. Suppose X is

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a compact oriented pseudomanifold of dimension $n = 4s$ with only isolated singularities and simply connected links. Considering the middle perversity intersection space $I^{\overline{m}}X$ gives us the isomorphism

$$\widetilde{H}I_r^{\overline{m}}(X) \cong \widetilde{H}I_{\overline{m}}^{4s-r}(X).$$

Then, it is shown that the two bilinear pairings over \mathbf{Q}

$$b_{HI} : \widetilde{H}I_{2s}^{\overline{m}}(X) \times \widetilde{H}I_{2s}^{\overline{m}}(X) \longrightarrow \mathbf{Q}$$

and

$$b_{IH} : IH_{2s}^{\overline{m}}(X) \times IH_{2s}^{\overline{m}}(X) \longrightarrow \mathbf{Q}$$

have the same Witt element in $W(\mathbf{Q})$.

It must be noticed that the pairing $\widetilde{H}I_{2s}^{\overline{m}}(X) \times \widetilde{H}I_{2s}^{\overline{m}}(X) \rightarrow \mathbf{Q}$ is not realized as the quadratic form associated to the generalized Poincaré duality of the space $I^{\overline{m}}X$. In fact, the space $I^{\overline{m}}X$ does not have a fundamental class. That is we can't express $b_{HI}(x, y)$ as $\langle [I^{\overline{m}}X], x \cup y \rangle$ where $[I^{\overline{m}}X]$ would be a fundamental class of the space $I^{\overline{m}}X$.

What we show is the existence of a rational Poincaré duality space $\mathcal{DP}(X)$. In particular the pairing defined above for the middle perversity intersection space can be realized as a pairing induced by a classical Poincaré duality and we have

$$b_{\mathcal{DP}(X)}(x, y) = \langle [\mathcal{DP}(X)], x \cup y \rangle.$$

Basically, we can separate the construction of the space $\mathcal{DP}(X)$ in two cases, whether the pseudomanifold X has only one isolated singularity or more.

Suppose given a compact, connected oriented pseudomanifold X of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ and denote by L_i the link of the singularity σ_i . We also suppose that the L_i are all simply connected.

We have the following definition and theorems

Definition 1.0.1. *Let X be a compact, connected pseudomanifold with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$. Denote by \overline{X} the normalization of X . A **good** rational Poincaré approximation of X is a topological space $\mathcal{DP}(X)$ such that*

- (1) $\mathcal{DP}(X)$ is a rational Poincaré duality space,
- (2) there is a rational factorization of the inclusion $i: X_{\text{reg}} \rightarrow \overline{X}$ in two maps

$$X_{\text{reg}} \xrightarrow{\phi} \mathcal{DP}(X) \xrightarrow{\psi} \overline{X}.$$

That is $\psi_r \circ \phi_r = i_r: H_r(X_{\text{reg}}) \rightarrow H_r(\overline{X})$ such that

- If $\dim X = 2s$, then
 - (a) $\phi_r: H_r(X_{\text{reg}}) \longrightarrow H_r(\mathcal{DP}(X))$ is an isomorphism for $2s - 1 > r > s$ and an injection for $r = s$,
 - (b) $\psi_r: H_r(\mathcal{DP}(X)) \longrightarrow H_r(\overline{X})$ is an isomorphism for $r < s$ and $r = 2s$.
- If $\dim X = 2s + 1$, then
 - (a) $\phi_r: H_r(X_{\text{reg}}) \longrightarrow H_r(\mathcal{DP}(X))$ is an isomorphism for $2s > r > s + 1$ and an injection for $r = s + 1$,
 - (b) $\psi_r: H_r(\mathcal{DP}(X)) \longrightarrow H_r(\overline{X})$ is an isomorphism for $r < s$ and $r = 2s + 1$ and a surjection for $r = s$.

We say that $\mathcal{DP}(X)$ is a **very good** rational Poincaré approximation of X if

- when $\dim X = 2s$, then ϕ_s is also an isomorphism.
- when $\dim X = 2s + 1$, then ϕ_{s+1} and ψ_s are also isomorphisms.

Theorem 1.1 (Unique isolated singularity case). *Let X be a compact, connected oriented normal pseudomanifold of dimension n with one isolated singularity of link L simply connected. There exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of*

X . Moreover if $\dim X \equiv 0 \pmod{4}$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X .

Depending of the dimension of X we use two different types of truncations on the link of the singularity L . When $\dim X = 2s$ this is the classical homology truncation defined by Markus Banagl in [2]. The odd dimensional case also use the Lagrangian truncation which will be defined in section 2.3.

The odd dimensional case breaks down in two subcases corresponding to the type of space we are dealing with. If the space is a Witt space we can construct a rational Poincaré duality space, and if the space is an L-space we can perform a Lagrangian truncation to get a rational Poincaré duality space. In the even dimensional case we can always construct a rational Poincaré duality space. We have the following theorem.

Theorem 1.2 (Multiple isolated singularities case). *Let X be a compact, connected oriented normal pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected. Then,*

- (1) *If $n = 2s$, there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X . Moreover, if $\dim X \equiv 0 \pmod{4}$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same as the Witt class associated to the middle intersection cohomology of X .*
- (2) *If $n = 2s + 1$ and X is either a Witt space or an L-space then there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X . Moreover when X is a Witt space $\mathcal{DP}(X)$ is a very good rational Poincaré approximation of X .*

The first section of this paper contains known definitions and results we will use. We first recall the definitions of pseudomanifolds, perversities and we give a brief account of rational homotopy theory. The second part is devoted to the theory of homological truncation theory and intersection spaces defined by Markus Banagl in [2], we also give a rational model of the intersection spaces in 2.3.3. We extend the homological truncation to a Lagrangian truncation in the third part 2.3.

The second section is devoted to the construction of the spaces $\mathcal{DP}(X)$. We first recall the notions we use about Poincaré duality. We then completely develop the method of construction, first with a unique isolated singularity and then explain how to modify the results to get the general theorem in the context of multiple isolated singularities.

We finish with a section of examples, the real algebraic varieties, the nodal hypersurfaces and the Thom spaces.

2. BACKGROUND, TRUNCATIONS AND INTERSECTION SPACES

2.1. Pseudomanifold, Goresky MacPherson perversity and rational homotopy theory. In this paper we are concerned with stratified pseudomanifold with isolated singularities of dimension n . That is a compact Hausdorff topological space X with stratification : $\emptyset \subset \Sigma \subset X_n = X$. Where $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ is a finite set of points, the isolated singularities. Each singularity has a cone-like open neighbourhood ${}^cL(\sigma_i, X)$ in X where $L(\sigma_i, X)$ is a topological manifolds of dimension $n - 1$. The space $L(\sigma_i, X)$ is called a link of σ_i in X . We will denote by $L_i := L(\sigma_i, X)$ a link in X of the singularity σ_i and by

$$L(\Sigma, X) := \bigsqcup_{\sigma_i \in \Sigma} L_i$$

the disjoint union of the links. The space $L(\Sigma, X)$ is then a disjoint union of topological manifolds of dimension $n - 1$.

Let X be a stratified pseudomanifold with only isolated singularities

$$\Sigma = \{\sigma_1, \dots, \sigma_\nu\}.$$

Removing a small open neighbourhood of each singularities σ_i gives us a manifold with boundary $(X_{reg}, \partial X_{reg})$ where the number of connected components of ∂X_{reg} is the number of connected components of $L(\Sigma, X)$. The manifold X_{reg} is called the regular part of X .

Definition 2.0.1. *Let X is a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$. We say that X is a normal pseudomanifold if the link L_i of each singularities σ_i is connected.*

Given X is a compact, connected oriented pseudomanifold of dimension n with only isolated singularities, one can construct its normalization \overline{X} by considering its regular part X_{reg} and by coning off separately each connected components of the boundary ∂X_{reg} . The space \overline{X} is then a normal pseudomanifold with only isolated singularities. We have a map $\overline{X} \rightarrow X$.

Since any topological pseudomanifold admits a normalization by the above process and since the definition 1.0.1 of a Poincaré duality approximation space of X involves a map from $\mathcal{DP}(X)$ to its normalization \overline{X} , we will always assume for the rest of this paper that the pseudomanifolds considered here are normal pseudomanifolds.

Since we only work with isolated singularities, a perversity is just a number $\overline{p} \in \{0, 1, \dots, \dim X - 2\}$. We denote by \overline{m} and \overline{n} the following perversities

$$\begin{cases} \overline{m} &:= \lfloor \frac{\dim X}{2} \rfloor - 1, \\ \overline{n} &:= \lceil \frac{\dim X}{2} \rceil - 1. \end{cases}$$

As for what concerns rational homotopy theory we refer to [7] and [6], we will denote by $A^*(K)$ some rational model of K and by $M(K)$ its minimal model. We will also use the two following results.

Theorem 2.1 (Rational Hurewicz theorem, [5], [11]). *Let K be a simply connected topological space with $\pi_i(K) \otimes \mathbf{Q} = 0$ for $1 < i < r$. Then the Hurewicz map induces an isomorphism*

$$\text{Hur}_i : \pi_i(K) \otimes \mathbf{Q} \longrightarrow H_i(K)$$

for $1 \leq i < 2r - 1$ and a surjection for $i = 2r - 1$.

Theorem 2.2 ([6, theorem 9.11 p.111]). *Every simply connected space K is rationally modelled by a CW-complex \tilde{K} for which the differential in the integral cellular chain complex is identically zero.*

Let's just say that the construction of $\varphi: \tilde{K} \rightarrow K$ is made inductively so that φ restrict to rational homotopy equivalences over some sub-CW-complex of K . The two following remarks follow from the proof.

- Remark 2.3.*
- (1) Let K_1 and K_2 be two simply connected CW-complexes such that $K_1^s = K_2^s$ for all $s \leq k$. Then $\tilde{K}_1^s = \tilde{K}_2^s$ for all $s \leq k$ and the map $\varphi_s^{K_1}$ and $\varphi_s^{K_2}$ are equal for all $s \leq k$.
 - (2) If K is a CW-complex of dimension n such that for the cellular chain complex $(C_*(K), \partial_*)$ we have $\ker \partial_n = 0$. Then \tilde{K} is a CW-complex of dimension $n - 1$.

2.2. Homological truncation and intersection spaces. In [2] Markus Banagl constructed, for a given perversity \bar{p} , a space called the perversity \bar{p} intersection space of X denoted by $I^{\bar{p}}X$. We briefly recall the construction.

Definition 2.3.1. *Given an integer $k \geq 3$, a (homological) k -truncation structure is a quadruple $(K, K/k, h, t_k K)$, where*

- K is a simply connected CW-complex,
- K/k is an k -dimensional CW-complex with $(K/k)^{k-1} = K^{k-1}$ and such that the group of k -cycles of K/k has a basis of cells,
- $h: K/k \rightarrow K^k$ is the identity on K^{k-1} and a cellular homotopy equivalence $\text{rel } K^{k-1}$, and
- $t_k K \subset K/k$ is a subcomplex such that

$$(1) \quad H_r(t_k K; \mathbf{Z}) \cong \begin{cases} H_r(K; \mathbf{Z}) & r < k, \\ 0 & r \geq k, \end{cases}$$

and such that $(t_k K)^{k-1} = K^{k-1}$.

Proposition 2.3.1 ([2]). *Given any integer $k \geq 3$, every simply connected CW-complex K can be completed to an k -truncation structure $(K, K/k, h, t_k K)$.*

For the construction of the quadruple $(K, K/k, h, t_k K)$ we send the reader to [2]. A similar construction will be explained later in this paper for the Lagrangian truncation.

Given any k -truncation structure $(K, K/k, h, t_k K)$, we have a homotopy class of maps $f: t_k K \rightarrow K$ given by the composition of the following maps

$$t_k K \hookrightarrow K/k \xrightarrow{h} K^k \hookrightarrow K$$

where the maps at the extremities are cellular inclusions.

Let now X be a compact, connected oriented normal pseudomanifold of dimension n with isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of simply connected links L_i , the L_i are then $(n-1)$ -dimensional compact manifolds.

Given a Goresky MacPherson perversity \bar{p} , put $k(\bar{p}) := n - 1 - \bar{p}$, we apply the $k(\bar{p})$ -truncation on each links L_i to get a family of CW-complexes $t_{k(\bar{p})} L_i$ together with homotopy classes of maps

$$f_i: t_{k(\bar{p})} L_i \rightarrow L_i.$$

We denote by

$$t_{k(\bar{p})} L(\Sigma, X) \xrightarrow{f} L(\Sigma, X)$$

the disjoint union of these maps, with $f := \sqcup_{\sigma_i} f_i$.

We define the two following homotopy cofibers.

First, consider the homotopy cofiber of the map f_i , which we denote by $t^{k(\bar{p})} L_i$ and call it the $k(\bar{p})$ -cotruncation of L_i . We have maps

$$f^i: L_i \longrightarrow t^{k(\bar{p})} L_i$$

and

$$H_r(t^{k(\bar{p})} L_i; \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & r = 0, \\ 0 & 1 \leq r < k(\bar{p}), \\ H_r(L_i; \mathbf{Z}) & r \geq k(\bar{p}). \end{cases}$$

We then have a family of maps

$$\partial X_{\text{reg}} = L(\Sigma, X) = \bigsqcup_i L_i \longrightarrow \bigsqcup_i t^{k(\bar{p})} L_i.$$

Then, we define by $t^{k(\bar{p})} L(\Sigma, X)$ to be the homotopy cofiber of the map f .

Definition 2.3.2. (1) *The intersection space $I^{\overline{P}}X$ of the space X is the homotopy pushout of the solid arrows diagram.*

$$\begin{array}{ccc} L(\Sigma, X) & \longrightarrow & X_{reg} \\ \downarrow & & \downarrow \\ t^{k(\overline{P})}L(\Sigma, X) & \dashrightarrow & I^{\overline{P}}X \end{array}$$

(2) *The normal intersection space $\mathcal{I}^{\overline{P}}X$ of the space X is the homotopy pushout of the solid arrows diagram.*

$$\begin{array}{ccc} L(\Sigma, X) & \longrightarrow & X_{reg} \\ \downarrow & & \downarrow \\ \bigsqcup_i t^{k(\overline{P})}L_i & \dashrightarrow & \mathcal{I}^{\overline{P}}X \end{array}$$

When X is normal with only one isolated singularity, there is no difference between the two definitions. Differences may arise only for the first homology group. In the first case, which is the original definition of [2], we have

$$H_1(I^{\overline{P}}X) = H_1(X_{reg}) \oplus \mathbf{Q}^{\beta_0(\partial X_{reg})-1}$$

where $\beta_0(\partial X_{reg})$ is the number of connected components of ∂X_{reg} . For the normal intersection space $\mathcal{I}^{\overline{P}}X$ we have

$$H_1(\mathcal{I}^{\overline{P}}X) = H_1(X_{reg}).$$

We now determine rational models of the truncation, cotruncation and the intersection space of X . Let L be a simply connected CW-complex of finite dimension and $(M(L), d)$ be its unique minimal Sullivan model.

Let us make some changes which will be useful when working rational models. Recall that L admits a cellular model \tilde{L} for which the differential in the integral cellular chain complex is identically zero by theorem 2.2. Denote by $\varphi: \tilde{L} \rightarrow L$ the rational homotopy equivalence given by this theorem.

Since $(t_k L)^{k-1} = L^{k-1}$, the first point of the remark 2.3 implies that

$$\widetilde{t_k L}^{k-1} = \tilde{L}^{k-1}.$$

By definition $t_k L$ is a CW-complex of dimension k such that for its cellular chain complex $(C_*(t_k L), \partial_*)$ we have $\ker \partial_k = 0$. The CW-complex $\widetilde{t_k L}$ is of dimension $k-1$ and is equal to \tilde{L}^{k-1} by the second point of the remark 2.3.

Consider the following diagram

$$\begin{array}{ccccccc} \tilde{L}^{k-1} & \xrightarrow{\quad \text{incl} \quad} & & & \tilde{L} & & \\ \varphi| \downarrow & & & & \downarrow \varphi & & \\ t_k L & \xrightarrow{\quad i \quad} & L/k & \xrightarrow{\quad h \quad} & L^k & \xrightarrow{\quad j \quad} & L \end{array}$$

where $\varphi|$ is the restriction of φ to the $(k-1)$ -cellular skeleton of \tilde{L} , which is then a rational homotopy equivalence. Since h is a homotopy equivalence relative to L^{k-1} this diagram is commutative. The fact that φ is a rational homotopy equivalences imply that the minimal models of L and \tilde{L} are isomorphic. The same is true for the minimal models of $t_k L$ and \tilde{L}^{k-1} .

Proposition 2.3.2. *For $k \geq 0$. Let $m : (\mathbf{M}(L), d) \rightarrow (A^*(L), d)$ be a rational model of L . Denote by \mathbf{C}_{k-1} a supplement of*

$$\ker(d^{k-1} : A^{k-1}(L) \rightarrow A^k(L))$$

and by I_{k-1} be the differential ideal of $A^(L)$ generated by $\mathbf{C}_{k-1} \oplus A^{\geq k}(L)$.*

A rational model of $t_k L$ is given by $A^(L)/I_{k-1}$ and a Sullivan representative of $f : t_k L \rightarrow L$ is given by the projection to the equivalence class*

$$A^*(L) \twoheadrightarrow A^*(L)/I_{k-1}.$$

Proof. By the discussion above, we have the isomorphism of minimal models

$$\mathbf{M}(\varphi) : (\mathbf{M}(L), d) \xrightarrow{\cong} (\mathbf{M}(\tilde{L}), d).$$

Composing its inverse with the quasi-isomorphism m gives us the following rational model of L

$$M := m \circ \mathbf{M}(\varphi)^{-1} : (\mathbf{M}(\tilde{L}), d) \rightarrow (A^*(L), d).$$

Consider now the map $\mathbf{M}(\text{incl}) : \mathbf{M}(\tilde{L}) \rightarrow \mathbf{M}(\tilde{L}^{k-1})$. By [7, theorem 2.28 p.66] there is a relative minimal model

$$\begin{array}{ccc} \mathbf{M}(\tilde{L}) & \xrightarrow{\mathbf{M}(\text{incl})} & \mathbf{M}(\tilde{L}^{k-1}) \\ \downarrow = & & \uparrow g \\ \mathbf{M}(\tilde{L}) & \xrightarrow{i} & \mathbf{M}(\tilde{L}) \otimes \wedge V \end{array}$$

where g is a quasi-isomorphism and i the canonical inclusion. The fact that $H^*(\mathbf{M}(\tilde{L})) = H^*(\mathbf{M}(\tilde{L}^{k-1})) = H^*(L)$ for $* \leq k-1$ implies that the elements of V are either of degree greater than or equal to k , or of degree $k-1$ and not in $\ker d^{k-1}$.

Let then I_{k-1} the differential ideal defined in the proposition and consider the following diagram.

$$\begin{array}{ccc} \mathbf{M}(\tilde{L}) & \xrightarrow{i} & \mathbf{M}(\tilde{L}) \otimes \wedge V \\ M \downarrow & & \downarrow \overline{M} \\ A^*(L) & \xrightarrow{p} & A^*(L)/I_{k-1} \end{array}$$

Where p is the projection map. We define the map \overline{M} by

$$\begin{cases} \overline{M}(a) = [M(a)] & a \in \mathbf{M}(\tilde{L}), \\ \overline{M}(\wedge V) = 0. \end{cases}$$

The image of a product is defined by

$$\begin{cases} \overline{M}(ab) = [M(a)M(b)] & a, b \in \mathbf{M}(\tilde{L}), \\ \overline{M}(vw) = 0 = \overline{M}(v)\overline{M}(w), & v, w \in V, \end{cases}$$

and for all $v \in V$ and all $a \in \mathbf{M}(\tilde{L})$, the degree of av is greater than or equal to k , so we define

$$\overline{M}(av) := 0 = \overline{M}(a)\overline{M}(v).$$

This diagram commutes and \overline{M} defines a quasi-isomorphism. \square

We now determine a rational model for the intersection spaces and the normal intersection spaces. The cotruncation being a homotopy cofiber, the next lemma follows from [6, Proposition 13.6]

Lemma 2.3.1. *The k -cotruncation of L , $t^k L$, being defined as the homotopy cofiber of the map $t_k L \rightarrow L$, a rational model is given by*

$$\mathbf{Q} \oplus \ker p = \mathbf{Q} \oplus I_{k-1}$$

where p is the map $p: A^*(L) \rightarrow A^*(L)/I_{k-1}$.

Proposition 2.3.3. *Let X be a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of simply connected links $L_i = L(\sigma_i, X)$. Let \bar{p} be a Goresky MacPherson perversity and*

$$\phi: (A^*(X_{reg}), d) \rightarrow (A^*(L(\Sigma, X)), d)$$

a surjective model of the inclusion $i: L(\Sigma, X) = \partial X_{reg} \rightarrow X_{reg}$. A rational model of $I^{\bar{p}}X$ is given by

$$AI_{\bar{p}}(X) := (A^*(X_{reg}), d) \oplus_{A^*(L)} (\mathbf{Q} \oplus I_{k(\bar{p})}, d)$$

where $(A^*(X_{reg}), d)$ is a rational model of the regular part of the pseudomanifold and $(\mathbf{Q} \oplus I_{k(\bar{p})}, d)$ a rational model of $t^{k(\bar{p})}L(\Sigma, X)$.

Proof. The intersection space $I^{\bar{p}}X$ of the space X is the homotopy pushout of the diagram.

$$\begin{array}{ccc} L(\Sigma, X) & \xrightarrow{i} & X_{reg} \\ \downarrow & & \downarrow \\ t^{k(\bar{p})}L(\Sigma, X) & \longrightarrow & I^{\bar{p}}X \end{array}$$

Then applying $A_{PL}(-)$ we have a diagram of pullback,

$$\begin{array}{ccc} A_{PL}(I^{\bar{p}}X) & \longrightarrow & A_{PL}(t^{k(\bar{p})}L(\Sigma, X)) \\ \downarrow & & \downarrow \\ A_{PL}(X_{reg}) & \xrightarrow{A_{PL}(i)} & A_{PL}(L(\Sigma, X)) \end{array}$$

and then the quasi isomorphism.

$$A_{PL}(I^{\bar{p}}X) \simeq A_{PL}(X_{reg}) \oplus_{A_{PL}(L(\Sigma, X))} A_{PL}(t^{k(\bar{p})}L(\Sigma, X)).$$

Given the rational models of X_{reg} , $L(\Sigma, X)$ and $t^{k(\bar{p})}L(\Sigma, X)$ thanks to the lemma 2.3.1, we get a map

$$(A^*(X_{reg}), d) \oplus_{A^*(L(\Sigma, X))} (\mathbf{Q} \oplus I_{k(\bar{p})}, d) \rightarrow A_{PL}(X_{reg}) \oplus_{A_{PL}(L(\Sigma, X))} A_{PL}(t^{k(\bar{p})}L(\Sigma, X)).$$

With the surjective model $\phi: (A^*(X_{reg}), d) \rightarrow (A^*(L(\Sigma, X)), d)$, we get a morphism of short exact sequences,

$$\begin{array}{ccccccc} \ker \phi & \longrightarrow & (A^*(X_{reg}), d_{X_{reg}}) \oplus_{A^*(L(\Sigma, X))} (\mathbf{Q} \oplus I_{k(\bar{p})}, d) & \longrightarrow & (\mathbf{Q} \oplus I_{k(\bar{p})}, d) \\ \downarrow & & \downarrow & & \downarrow \\ \ker A_{PL}(i) & \longrightarrow & A_{PL}(X_{reg}) \oplus_{A_{PL}(L(\Sigma, X))} A_{PL}(t^{k(\bar{p})}L(\Sigma, X)) & \longrightarrow & A_{PL}(t^{k(\bar{p})}L(\Sigma, X)) \end{array}$$

The result follows from an application of the five lemma. \square

Proposition 2.3.4. *Let X be a compact, connected oriented pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of simply connected links $L_i = L(\sigma_i, X)$. Let \bar{p} be a Goresky MacPherson perversity and*

$$\phi: (A^*(X_{reg}), d) \rightarrow (A^*(L(\Sigma, X)), d)$$

a surjective model of the inclusion $i: L(\Sigma, X) = \partial X_{reg} \rightarrow X_{reg}$. A rational model of the normal intersection space $\mathcal{I}^{\overline{p}}X$ is given by

$$A\mathcal{I}_{\overline{p}}(X) := (A^*(X_{reg}), d) \oplus_{A^*(L)} \left(\bigoplus_i \mathbf{Q} \oplus I_{k(\overline{p}, i)}, d \right)$$

where $(A^*(X_{reg}), d)$ is a rational model of the regular part of the pseudomanifold and $(\mathbf{Q} \oplus I_{k(\overline{p}, i)}, d)$ a rational model of $t^{k(\overline{p})}L_i$.

Proof. The proof is exactly the same as the previous proposition unless the normal intersection space $\mathcal{I}^{\overline{p}}X$ is the homotopy pushout of the following diagram.

$$\begin{array}{ccc} L(\Sigma, X) & \xrightarrow{i} & X_{reg} \\ \downarrow & & \downarrow \\ \bigsqcup_i t^{k(\overline{p})}L_i & \longrightarrow & \mathcal{I}^{\overline{p}}X \end{array}$$

□

In the odd dimensional case, there is a class of pseudomanifolds X for which the truncations $t_{k(\overline{m})}$ and $t_{k(\overline{n})}$ will coincide.

Definition 2.3.3. Let X be a compact, connected oriented pseudomanifold of dimension $n = 2s + 1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. X is a Witt space if $H^s(L_i) = 0$ for all $\sigma_i \in \Sigma$.

Example 2.1. (1) The suspension of the complex projective space SCP^3 is a Witt space since $H^3(\mathbf{CP}^3) = 0$.
 (2) The suspension of the complex projective plane SCP^2 is not a Witt-space since $H^2(\mathbf{CP}^2) = \mathbf{Q}$.

2.3. Lagrangian truncation and Lagrangian intersection spaces. First, we recall some facts about quadratic spaces that we will need later.

Definition 2.3.4. A regular quadratic space (E, b) is a vector space of finite dimension E together with a non degenerate bilinear form

$$b: E \times E \rightarrow \mathbf{k},$$

b being either be a symmetric form or an skew-symmetric one.

Definition 2.3.5. An isotropic subspace V of (E, b) is a subspace of E such that for all $x \in V$, $q(x) = b(x, x) = 0$. If $2 \dim V = \dim E$, V is then called a Lagrangian subspace.

Theorem 2.4 ([12], Hyperbolic completion). Let (E, b) be a regular quadratic space of dimension $2m$ and suppose that E posses a Lagrangian subspace V of dimension m . Then there exist a basis

$$(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$$

of E such that

$$\begin{cases} b(a_i, a_j) &= 0, \\ b(a_i^*, a_j^*) &= 0, \\ b(a_i, a_j^*) &= \delta_{ij}. \end{cases}$$

In particular $(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$ is a basis in the usual sense of E . We also call this an hyperbolic basis.

The spaces generated respectively by (a_1, \dots, a_m) and (a_1^*, \dots, a_m^*) are then Lagrangian subspaces.

Consider K as a simply connected n -dimensional CW-complex satisfying Poincaré duality with $n = 2s$. We denote by b the non degenerate bilinear form induced by the Poincaré duality with \mathbf{Q} coefficients, consider $\dim H^s(K) = 2m$ and

$$b : H^s(K) \times H^s(K) \longrightarrow \mathbf{Q}$$

where $b(x, y) := \langle x \cup y, [K] \rangle$ with $[K] \in H_{2s}(K)$ the fundamental class and $\langle -, - \rangle$ the evaluation form.

If b is symmetric suppose that $H^s(K)$ posses a Lagrangian subspace V of dimension m , let then (a_1, \dots, a_m) be a basis of V and thanks to the theorem 2.4, complete (a_1, \dots, a_m) into a hyperbolic basis $(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$ of $H^s(K)$.

If b is skew-symmetric then there always exists a Lagrangian subspace and thanks to the theorem 2.4, there exist a hyperbolic basis $(a_1, \dots, a_m, a_1^*, \dots, a_m^*)$ of $H^s(K)$.

Either way, denote by V and V^* the subspaces respectively generated by

$$V := \langle a_1, \dots, a_m \rangle$$

and

$$V^* := \langle a_1^*, \dots, a_m^* \rangle.$$

We have $H^s(K) = V \oplus V^*$.

Remark that since $b(a_i, a_i^*) = 1$, a_i and a_i^* are Poincaré duals to each other. Denote by \overline{V} and \overline{V}^* the Poincaré duals in $H_s(K)$ of respectively V and V^* and by $(\overline{a}_1, \dots, \overline{a}_m)$ the basis of \overline{V} and by $(\overline{a}_1^*, \dots, \overline{a}_m^*)$ the basis of \overline{V}^* . We have the direct sum

$$H_s(K) = \overline{V} \oplus \overline{V}^*.$$

Applying theorem 2.2 to K we have a rational cellular homotopy equivalence $\varphi: \tilde{K} \longrightarrow K$ where $(C_*(\tilde{K}), 0)$ is the integral cellular chain complex. We perform and define the Lagrangian truncation on \tilde{K} .

Since the differential of $C_*(\tilde{K})$ is zero we have

$$C_s(\tilde{K}) \otimes \mathbf{Q} = H_s(\tilde{K}) \cong H_n(K) = \overline{V} \oplus \overline{V}^*$$

By simply connectivity and the Hurewicz theorem we have the isomorphism

$$\text{Hur}_s : \overline{V} \oplus \overline{V}^* \xrightarrow{\cong} \pi_s(\tilde{K}^s, \tilde{K}^{s-1}).$$

Let

$$\lambda_i, \lambda_i^* : S^{s-1} \rightarrow \tilde{K}^{s-1}$$

be choices of representatives for the homotopy classes of $d\overline{a}_i, d\overline{a}_i^*$. Then for s -cells $\{t_i\}, \{t_i^*\}$ and using λ_i, λ_i^* as attaching maps we define

$$\tilde{K}/\mathcal{L} := \tilde{K}^{s-1} \cup \bigcup_{\lambda_i} t_i \cup \bigcup_{\lambda_i^*} t_i^*.$$

We get the cellular homotopy equivalence $h: \tilde{K}/\mathcal{L} \rightarrow \tilde{K}^s$ rel \tilde{K}^{s-1} the same way as the classical spatial homology truncation in [2, proposition 1.6].

Definition 2.4.1. *The Lagrangian truncation of the CW-complex K is defined by*

$$t_{\mathcal{L}}K := \tilde{K}^{s-1} \cup \bigcup_{\lambda_i} t_i^*.$$

Moreover, we have $h_s: H_s(\tilde{K}/\mathcal{L}) \cong H_s(\tilde{K}) \cong H_s(\tilde{K}^s, \tilde{K}^{s-1})$. If $(C_*(\tilde{K}/\mathcal{L}), \partial)$ denotes the integral cellular chain complex of \tilde{K}/\mathcal{L} this implies that

$$(C_{\leq s}(\tilde{K}/\mathcal{L}), \partial) \otimes \mathbf{Q} \longrightarrow (C_s(\tilde{K}), 0) \otimes \mathbf{Q}$$

yields an isomorphism of homology in degree s . This implies that

$$\partial: C_s(\tilde{K}/\mathcal{L}) \rightarrow C_{s-1}(\tilde{K}/\mathcal{L})$$

is zero.

The comparison map $t_{\overline{\mathcal{L}}}K \rightarrow K$ is defined as the composition of the maps

$$t_{\overline{\mathcal{L}}}K \hookrightarrow \tilde{K}/\mathcal{L} \xrightarrow{h} \tilde{K}^s \hookrightarrow \tilde{K} \xrightarrow{\varphi} K$$

where the arrows \hookrightarrow denote cellular inclusions, h is a cellular homotopy equivalence rel \tilde{K}^{s-1} and φ a cellular rational homotopy equivalence. We have

$$(2) \quad H_r(t_{\overline{\mathcal{L}}}K) \cong \begin{cases} H_r(K) & r \leq s-1 \\ \overline{V^*} & r = s \\ 0 & r > s. \end{cases}$$

We now define, with the help of Lagrangian truncation, the space called the Lagrangian intersection space associated to X

Let X a compact, connected oriented normal pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. This time we fix the dimension of X to be odd, $n = 2s + 1$.

Denote by

$$b_i: H^s(L_i) \times H^s(L_i) \longrightarrow \mathbf{Q}$$

the non degenerate bilinear form induced by Poincaré duality on the links L_i . Suppose that for all i , $H^s(L_i)$ admit a Lagrangian subspace V_i with respect to the bilinear form b_i . To each L_i we apply the Lagrangian truncation process to get maps

$$f_i: t_{\overline{\mathcal{L}}}L_i \longrightarrow L_i.$$

Denote by $t^{\overline{\mathcal{L}}}L_i$ the homotopy cofiber of the map f_i and call it the Lagrangian cotruncation of L_i , we then have a map

$$f^i: L_i \longrightarrow t^{\overline{\mathcal{L}}}L_i.$$

And

$$H_r(t^{\overline{\mathcal{L}}}L_i) \cong \begin{cases} \mathbf{Q} & r = 0, \\ 0 & 1 \leq r < s, \\ \overline{V_i} & r = s, \\ H_r(K) & s+1 \leq r \leq 2s. \end{cases}$$

Definition 2.4.2. The Lagrangian intersection space $\mathcal{I}^{\overline{\mathcal{L}}}X$ of the space X is the homotopy pushout of the solid arrows diagram.

$$\begin{array}{ccc} L(\Sigma, X) & \longrightarrow & X_{reg} \\ \downarrow & & \vdots \\ \bigsqcup_i t^{\overline{\mathcal{L}}}L_i & \dashrightarrow & \mathcal{I}^{\overline{\mathcal{L}}}X \end{array}$$

We want to know when we can perform Lagrangian truncation to get Lagrangian intersection spaces. Lets us first define the class of spaces for which it is possible.

Definition 2.4.3. Let X be a compact, connected oriented normal pseudomanifold of dimension $2s + 1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. X is an L -space if $H^s(L_i)$ has a Lagrangian subspace with respect to the non degenerate bilinear form $b_i: H^s(L_i) \times H^s(L_i) \rightarrow \mathbf{Q}$ for all $\sigma_i \in \Sigma$.

Example 2.2. (1) The suspension of the torus ST^2 is an L-space since

$$H^1(L) = H^1(T^2) = \mathbf{Q} \oplus \mathbf{Q}.$$

(2) The suspension of the complex projective plane SCP^2 is not an L-space since $H^2(\mathbf{CP}^2) = \mathbf{Q}$.

We have the following criterion saying when a space X of odd dimension is an L-space and thus when we can perform a Lagrangian truncation. Let L_i be a link of a singularity of X , L_i is a connected compact manifold of even dimension. Consider its non-degenerate bilinear form induced by Poincaré duality and denote it by b_i and let $\sigma(b_i)$ be its reduced signature, $\sigma(b_i)$ is related to the Pontryagin numbers by the Hirzebruch signature formula.

Proposition 2.4.1. *Let X a compact, connected oriented normal pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ of links L_i simply connected. Then*

- (1) *If $n = 4s + 3$, X is an L-space.*
- (2) *If $n = 4s + 1$, X is an L-space if and only if $\sigma(b_i) = 0$ for all i .*

Proof. Suppose first that $\dim L_i \equiv 2 \pmod{4}$, that is $\dim X = n = 4s + 3$. In that case the non-degenerate bilinear form induced by Poincaré duality

$$b_i: H^{2s+1}(L_i) \times H^{2s+1}(L_i) \longrightarrow \mathbf{Q}$$

is skew-symmetric due to the graded commutativity. The form b_i is then a non-degenerate symplectic form for all $\sigma_i \in \Sigma$, we always have a Lagrangian subspace in that case.

Now, if $\dim L_i \equiv 0 \pmod{4}$, that is $\dim X = 4s + 1$. We can't always apply the Lagrangian truncation. The existence of a Lagrangian subspace for $H^{2s}(L_i)$ is given by the theorem of Sullivan and Barge, see [3] and [15], about rational classification of simply connected manifolds.

If X is an L-space it is clear by definition 2.4.3 that $\sigma(b_i) = 0$ for all i .

On the other hand, let $(\wedge V_i, d)$ be a Sullivan model of L_i and $p = \{p_j\}$ the Pontryagin numbers of L_i related to $\sigma(b_i) = 0$. Obviously L_i realizes the pair $((\wedge V_i, d), p)$, then by the Sullivan-Barge theorem this implies the quadratic form on $H^{2s}(L_i)$ is equivalent over \mathbf{Q} to a quadratic form

$$\sum_{k=1}^m x_k^2 - \sum_{k'=1}^m x_{k'}^2.$$

This quadratic form is then hyperbolic and posses a Lagrangian subspace. \square

3. THE CONSTRUCTION OF POINCARÉ DUALITY SPACES

3.1. Poincaré duality. We recall here the definitions and results about Poincaré duality needed in the rest of the paper.

Definition 3.0.4. *Let (X, Y) be a pair of CW-complex, we say that (X, Y) is a rational Poincaré duality pair of dimension n if :*

- (1) *$\dim_{\mathbf{Q}} H_r(X; \mathbf{Q})$ is finite for all r ,*
- (2) *Y is a sub-CW-complex of X with the same property,*
- (3) *there exists a class $[x] \in H_n(X, Y; \mathbf{Q})$ such that*

$$- \cap [x] : H^r(X; \mathbf{Q}) \longrightarrow H_{n-r}(X, Y; \mathbf{Q})$$

is an isomorphism. We call $[x]$ an orientation class of (X, Y)

Remark 3.1. Let (X, Y) be a Poincaré duality pair of dimension n , then $Y = (Y, \emptyset)$ is a Poincaré duality pair of dimension $n - 1$. Indeed, if $[x] \in H_n(X, Y)$ is an orientation class, then $[y] = \partial[x] \in H_{n-1}(Y)$ is an orientation class of Y . We say that Y is a Poincaré complex of dimension $n - 1$. We also say that $(Y, [y])$ is the oriented boundary of $(X, Y, [x])$.

Let $(X_1, Y, [x_1])$ and $(X_2, Y, [x_2])$ be two oriented Poincaré duality pairs of dimension n with the same oriented boundary $(Y, [y])$. Let $\hat{X} := X_1 \cup_Y X_2$ the CW-complex obtained by glueing X_1 and X_2 along their common boundary Y .

Theorem 3.2 ([4], Glueing of 2 oriented Poincaré duality pairs). *Consider the following diagram*

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & H_n(\hat{X}) \\ & & \downarrow i \\ H_n(X_1, Y) \oplus H_n(X_2, Y) & \xrightarrow{i_1 \oplus i_2} & H_n(\hat{X}, Y) \\ & & \downarrow \partial \\ & & H_{n-1}(Y) \end{array}$$

and let $[\hat{x}] = i^{-1}(i_1 \oplus i_2([x_1], [-x_2])) \in H_n(\hat{X})$. Then any two of the following conditions imply the third.

- (1) $(\hat{X}, [\hat{x}])$ is an oriented Poincaré complex of dimension n without boundary,
- (2) $(Y, [y])$ is a Poincaré complex of dimension $n - 1$ with orientation class $[y] = \partial_1[x_1] = \partial_2[x_2] = \partial_0[\hat{x}] \in H_{n-1}(Y)$,
- (3) $(X_i, Y, [x_i])$ are Poincaré duality pairs of dimension n with orientation classes $[x_i] \in H_n(X_i, Y)$.

Let $(X, Y, [x])$ be an oriented Poincaré duality pair of dimension $n = 4s$, the following diagram

$$\begin{array}{ccccc} H^{2s}(X, Y) & \longrightarrow & H^{2s}(X) \\ \downarrow - \cap [x] \cong & & \downarrow \cong \\ H_{2s}(Y) & \xrightarrow{i_*} & H_{2s}(X) & \longrightarrow & \text{hom}(H_{2s}(X), \mathbf{Q}) \end{array}$$

gives the vector space $H_{2s}(X)$ a symmetric bilinear form of kernel $i_*(H_{2n}(Y))$ denoted by b_X .

Lemma 3.2.1 (Novikov). *Let $(X_1, Y, [x_1])$ and $(X_2, Y, [x_2])$ 2 oriented Poincaré duality pairs of dimension $n = 4s$ with the same oriented boundary $(Y, [y])$. If $(\hat{X}, [\hat{x}])$ is the space obtained by glueing as in theorem 3.2, then*

$$[b_{\hat{X}}] = [b_{X_1}] - [b_{X_2}] \text{ in } W(\mathbf{Q}).$$

The rational homotopy type of rational Poincaré duality spaces does not depend of the fundamental class. As stated in the following theorem.

Theorem 3.3 ([13]). *Let H be a Poincaré duality algebra of top dimension n and $H^1 = 0$. Let X be a simply connected rational space with $H(X) \cong H$ except $H^n(X) = 0$. If $Y = X \cup e^n$ with $H(Y) \cong H$, then the rational homotopy type of Y is determined by X . Moreover, the cell e^n is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(X) \otimes \mathbf{Q}$.*

We refer to [6, 13.e p.175] for the definition of a Whitehead product.

3.2. The unique isolated singularity case. In this part we prove the following theorem

Theorem 3.4 (Unique isolated singularity case). *Let X be a compact, connected oriented normal pseudomanifold of dimension n with one isolated singularity of link L simply connected. Then, there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X . Moreover if $\dim X \equiv 0 \pmod{4}$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X .*

3.2.1. The even dimensional case. Consider now X a compact, connected oriented normal pseudomanifold of dimension $n = 2s$ with one isolated singularity σ of simply connected link L . Since $\overline{m} = \overline{n}$ we have a well defined intersection space $IX := I^{\overline{m}}X = I^{\overline{n}}X$.

Let $\phi : S^{2s-1} \rightarrow tL$ be an arbitrary continuous map with $tL := t^{k(\overline{m})}L$ the middle cotruncation of the link. We denote by $t^\phi L$ the space obtained as the result of the following homotopy pushout :

$$\begin{array}{ccc} S^{2s-1} & \hookrightarrow & D^{2s} \\ \phi \downarrow & \text{HPO} & \downarrow \\ tL & \dashrightarrow & t^\phi L \end{array}$$

Lemma 3.4.1. *$(t^\phi L, L)$ is a Poincaré duality pair if and only if ϕ_{2s-1} is an isomorphism. Where ϕ_{2s-1} is the connecting homomorphism*

$$\phi_{2s-1} : H_{2s}(t^\phi L, L) \longrightarrow H_{2s-1}(L)$$

in the long exact sequence of the pair $(t^\phi L, L)$ induced by the attaching map ϕ .

Proof. Suppose $(t^\phi L, L)$ is a Poincaré duality pair and denote by $[e_\phi]$ a choice of orientation class for the pair. By definition we have $\partial_{2s}[e_\phi] = [L]$, but $\partial_{2s} = \phi_{2s-1}$.

On the other hand, if $\phi_{2s-1} : H_{2s}(t^\phi L, L) \cong H_{2s-1}(L)$ is an isomorphism let us denote by $[e_\phi] := \phi_{2s-1}^{-1}([L])$. We have to check the commutativity of the following square for the different values of r

$$\begin{array}{ccc} H^r(t^\phi L) & \xrightarrow{- \cap [e_\phi]} & H_{2s-r}(t^\phi L, L) \\ \text{incl}^* \downarrow & & \downarrow \partial_{2s-r} \\ H^r(L) & \xrightarrow[-\cap [L]]{\cong} & H_{2s-1-r}(L) \end{array}$$

and the fact that this induces an isomorphism on the upper row. Which are straightforward calculations. \square

We now look at a condition on ϕ to be an isomorphism.

Lemma 3.4.2. *ϕ_{2s-1} is an isomorphism if and only if*

$$\text{HUR}_{2s-1}([\phi]) \neq 0$$

in $H_{2s-1}(tL) = H_{2s-1}(L) = \mathbf{Q}$.

Since the pair $(X_{\text{reg}}, \partial X_{\text{reg}})$, with $\partial X_{\text{reg}} = L$, satisfies Poincaré-Lefschetz duality this is a Poincaré duality pair with the same boundary that $(t^\phi L, L)$. The space

$$\mathcal{DP}(X) := X_{\text{reg}} \cup_L t^\phi L$$

is then a Poincaré complex of dimension $2s$ without boundary and of orientation class given by

$$[\mathcal{DP}(X)] = i^{-1}(i_1 \oplus i_2([X_{reg}, L], [-e_\phi]))$$

We now show the relation between IX and $\mathcal{DP}(X)$.

Proposition 3.4.1. *Let X be a compact, connected oriented normal pseudomanifold of dimension $n = 2s$ with one isolated singularity of link L simply connected. The space $\mathcal{DP}(X)$ is then rationally homotopy equivalent to $IX \cup e^{2s}$. If moreover $H^1(X_{reg}) = 0$, then e^{2s} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(IX) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by IX .*

Proof. Consider the following diagram.

$$\begin{array}{ccccc} & & L = \partial X_{reg} & \xleftarrow{i_1} & X_{reg} \\ & & \downarrow & & \downarrow \\ S^{2s-1} & \xrightarrow{\phi} & tL & \xrightarrow{\quad (I) \text{ HPO} \quad} & IX \\ i_2 \downarrow & (III) \text{ HPO} & \downarrow & (II) & \downarrow \\ D^{2s} & \xrightarrow{\quad} & t^\phi L & \xrightarrow{\quad} & \mathcal{DP}(X) \end{array}$$

The square (I) is a homotopy pushout by definition of the construction of the space IX , since i_1 is a cofibration $\mathcal{DP}(X)$ is the homotopy pushout of the diagram $tL_\phi \leftarrow tL \leftarrow L \hookrightarrow M$. By the properties of successive homotopy pushouts (see for exemple [14]) the square (II) is a homotopy pushout. The square (III) is also a homotopy pushout by definition of $t^\phi L$, this implies that the outside square (II) + (III) is a homotopy pushout. We have the commutative square

$$\begin{array}{ccc} S^{2s-1} & \xrightarrow{\quad} & IX \\ \downarrow & & \downarrow \\ D^{2s} & \xrightarrow{\quad} & \mathcal{DP}(X) \end{array}$$

which is then a homotopy pushout.

So we have a rational homotopy equivalence between $\mathcal{DP}(X)$ and $IX \cup e^{2s}$.

Suppose now that we also have $H^1(X_{reg}) = 0$. The space $\mathcal{DP}(X)$ is simply connected and the theorem 3.3 then tells us how e^{2s} is attached to IX . \square

In the case of a pseudomanifold of dimension $n = 4s$ with isolated singularities, Markus Banagl showed in [2, theorem 2.28] that the intersection form

$$b_{HI}: \tilde{H}I_{2s}^{\overline{m}}(X) \otimes \tilde{H}I_{2s}^{\overline{m}}(X) \longrightarrow \mathbf{Q}$$

has the same Witt element that the Goresky-MacPherson intersection form

$$b_{IH}: IH_{2s}^{\overline{m}}(X) \otimes IH_{2s}^{\overline{m}}(X) \longrightarrow \mathbf{Q}.$$

that is $[b_{HI}] = [b_{IH}] \in W(\mathbf{Q})$, where $W(\mathbf{Q})$ is the Witt group of the rationals.

Applying the lemma 3.2.1 to the Poincaré duality pair $(t^\phi L, L)$ constructed above shows that this pair is endowed with a symmetric bilinear form of kernel $i_*(H_{2s}(L)) = H_{2s}(t^\phi L)$, that is the form is the zero form. We see that the Witt class of the intersection form $[b_{\mathcal{DP}(X)}]$ of $\mathcal{DP}(X)$ is completely determined by the intersection form of the regular part $(X_{reg}, \partial X_{reg})$. Which is also the case of IX as showed in [2, theorem 2.28]. Therefore we have the following corollary.

Corollary 3.4.1. *$\mathcal{DP}(X)$ is a Poincaré duality rational space whose Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X .*

3.2.2. *The odd dimensional case.* Consider now X a compact, oriented pseudomanifold of dimension $n = 2s + 1$ with one isolated singularity σ of simply connected link L .

First let us consider that X is a Witt space, that is $H_s(L) = 0$. We then have the following proposition.

Proposition 3.4.2. *Let X be a compact, connected oriented normal pseudomanifold of dimension $n = 2s + 1$ with one isolated singularity σ of simply connected link L . Suppose moreover that X is a Witt space. The constructions of the even dimensional case extend to this case and there exists a rational Poincaré approximation $\mathcal{DP}(X)$ of X .*

Proof. We have to show that there exists a cotruncation tL of the link of the singularity and a map $\phi \in \pi_{2s}(tL) \otimes \mathbf{Q}$ such that the pair $(t^\phi L, L)$ is a Poincaré duality pair. If so, the theorem 3.2 and the proposition 3.4.1 can be applied.

Consider the cotruncation $t^{k(\bar{n})}L$ given by the upper middle perversity \bar{n} . By definition of Witt spaces and of the cotruncation we have

$$H_s(t^{k(\bar{n})}L) = H_s(L) = 0.$$

So in fact $t^{k(\bar{n})}L$ is s -connected and we have

$$t^{k(\bar{n})}L = t^{k(\bar{n})}L := tL.$$

By the rational Hurewicz theorem we have the isomorphism

$$\pi_{2s}(tL) \otimes \mathbf{Q} \xrightarrow{\cong} H_{2s}(tL).$$

We still denote by ϕ the map obtained by this isomorphism, ϕ_{2s} as in the lemma 3.4.1 is then a isomorphism and the pair $(t^\phi L, L)$ is a Poincaré duality pair. \square

Every pseudomanifold X of dimension $n = 2s + 1$ which is not a Witt space is in fact an L -space due to the following result of Thom.

Lemma 3.4.3 (Thom, [10]). *Let $(X, [x])$ be a rational Poincaré complex of dimension $n = 4s$ such that $(X, [x])$ is the boundary of a rational Poincaré duality pair $(Y, [y])$. Then $[b_X] = 0 \in W(\mathbf{Q})$.*

The link L of the singularity of X is a manifold of dimension $2s$ and we have the non degenerate bilinear form induced by Poincaré duality.

$$b_L : H^s(L) \times H^s(L) \longrightarrow \mathbf{Q}$$

Suppose that X is not a Witt space. By this result of Thom, the Witt class $[b_L] \in W(\mathbf{Q})$ of the intersection form associated to L is zero. This implies that b_L is hyperbolic and we have the existence of a Lagrangian subspace.

Corollary 3.4.2. *Any compact, connected oriented normal pseudomanifold of dimension $4s + i$, $i = 1, 3$, with one isolated singularity σ of simply connected link L is an L -space.*

Let us then fix X a compact oriented L -space of dimension $4s + i$, $i = 1, 3$, with one isolated singularity σ of simply connected link L . The link L of the singularity is of dimension $4s + i - 1$ and thanks to the lemma 3.4.3 we have

$$H^{2s+\frac{i-1}{2}}(L) = V \oplus V^*$$

where V is a Lagrangian subspace of dimension $\frac{1}{2} \dim H^{2s+1}(L) := m$.

Suppose (a_1, \dots, a_m) is a basis of V and complete it into a hyperbolic basis using theorem 2.4. Apply then the Lagrangian truncation to L and denote by $t^{\overline{\mathcal{L}}}L$ the homotopy cofiber of the map $t_{\overline{\mathcal{L}}}L \rightarrow L$.

Let $(M(t^{\overline{\mathcal{L}}}L), d)$ be the Sullivan minimal model of this Lagrangian cotruncation and let ϖ be the element of degree $4s + i - 1$ in $(M(t^{\overline{\mathcal{L}}}L), d)$ representing the fundamental class $[L]^* \in H^{4s+i-1}(L)$.

Proposition 3.4.3. *Suppose ϖ is an indecomposable element of $(M(t^{\overline{\mathcal{L}}}L), d)$, then there exists a map $\phi \in \pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q}$ such that $\text{HUR}_{4s+i-1}([\phi]) = 1$.*

Proof. Suppose ϖ is an indecomposable element, that is $\varpi \in W^{4s+i-1}$ where $W = \bigoplus_{k \geq 0} W^k$ is the graded vector space generating $M(t^{\overline{\mathcal{L}}}L)$. Then by [6, theorem 15.11], we have the natural isomorphism

$$W^{4s+i-1} \xrightarrow{\cong} \text{hom}(\pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L), \mathbf{Q}).$$

That natural isomorphism is the same as saying that the bilinear pairing

$$\langle -, - \rangle : W \times \pi_*(t^{\overline{\mathcal{L}}}L) \rightarrow \mathbf{Q}$$

is non degenerate. So there is a map $\varphi \in \pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L)$ such that $\langle \varpi, [\varphi] \rangle \neq 0$.

Recall that

$$m_{t^{\overline{\mathcal{L}}}L} : (\wedge W, d) = (M(t^{\overline{\mathcal{L}}}L), d) \rightarrow A_{\text{PL}}(t^{\overline{\mathcal{L}}}L)$$

denotes the minimal Sullivan model of $t^{\overline{\mathcal{L}}}L$ and denote by

$$\text{HUR}_k : \pi_k(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q} \rightarrow H_k(t^{\overline{\mathcal{L}}}L)$$

the Hurewicz map. Since $\text{im } d \subset \wedge^{\geq 2} W$, quotienting by $\wedge^{\geq 2} W$ defines a linear map $\xi : H^+(\wedge W) \rightarrow W$, since ϖ represents the fundamental class $[L]^* \in H^{4s+i-1}(L) = H^{4s+i-1}(t^{\overline{\mathcal{L}}}L)$, we clearly have a element $[\varpi] \in H^+(\wedge W)^{4s+i-1}$ such that $\xi([\varpi]) = \varpi$.

Denote by $\{-, -\}$ the bilinear pairing between cohomology and homology defined by $\{[f], [c]\} := f(c)$, since we work on \mathbf{Q} that pairing is also non degenerate. By the definition of the pairing $\langle -, - \rangle$ (see [6, p 172-173]) we have

$$\langle \xi([\varpi]), [\varphi] \rangle = \{H(m_{t^{\overline{\mathcal{L}}}L})[\varpi], \text{HUR}_{4s+i-1}(\varphi)\} \neq 0.$$

Since $H(m_{t^{\overline{\mathcal{L}}}L})[\varpi] = [L]^*$, we have $\text{HUR}_{4s+i-1}(\varphi) = q[L]$ with $q \in \mathbf{Q} - \{0\}$. Then

$$\phi := \frac{1}{q}\varphi \in \pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q}$$

is the map we wanted. □

Lemma 3.4.4. *ϖ is an indecomposable element of $(M(t^{\overline{\mathcal{L}}}L), d)$, that is*

$$\varpi \in W^{4s+i-1}$$

where $W = \bigoplus_{k \geq 0} W^k$ is the graded vector space generating $M(t^{\overline{\mathcal{L}}}L)$.

Proof. The elements of degree $2s + \frac{i-1}{2}$ of $(M(t^{\overline{\mathcal{L}}}L), d)$ come from the Lagrangian V so for all elements $x, y \in M(t^{\overline{\mathcal{L}}}L)^{2n + \frac{i-1}{2}}$ there exists an element $z \in M(t^{\overline{\mathcal{L}}}L)^{4n+i-2}$ such that $dz = x \cdot y$, in particular, none of these products are equal to ϖ . For degree reasons these were the only elements we had to care about. □

Denote by ϕ the element of $\pi_{4s+i-1}(t^{\overline{\mathcal{L}}}L) \otimes \mathbf{Q}$ obtained by this process, like in the general case, and consider the homotopy pushout.

$$\begin{array}{ccc} S^{4s+i-1} & \hookrightarrow & D^{4s+i} \\ \phi \downarrow & \text{HPO} & \downarrow \\ t^{\overline{\mathcal{L}}}L & \longrightarrow & t^{\overline{\mathcal{L}},\phi}L \end{array}$$

Proposition 3.4.4. *$(t^{\overline{\mathcal{L}},\phi}L, L)$ is a Poincaré duality pair if and only if ϕ_{4s+i-1} is an isomorphism.*

We denote by $\mathcal{DP}(X)$ the space obtained by the glueing of the two Poincaré duality pairs $(X_{reg}, \partial X_{reg}) = (X_{reg}, L)$ and $(t^{\overline{\mathcal{L}}}L_\phi, L)$ following the theorem 3.2. We have the last part of the theorem 1.1.

Proposition 3.4.5. *Let X be a compact, connected oriented normal pseudomanifold of dimension $n = 4s + 1$ or $n = 4s + 3$ with one isolated singularity σ of simply connected link L . Suppose moreover that X is an L -space. Then there exists a rational Poincaré approximation $\mathcal{DP}(X)$ of X .*

Just like in the even dimensional case with the proposition 3.4.1. We can relate the spaces $\mathcal{DP}(X)$ to the intersection and Lagrangian intersections spaces, and get more precision on how to attach the top cell in the simply connected case. When X is a Witt space, we denote by $IX = I^m X = I^{\overline{m}} X$.

Proposition 3.4.6. *Let X be a compact, connected oriented normal pseudomanifold of dimension $n = 2s+1$ with one isolated singularity of link L simply connected.*

- (1) *Suppose X is a Witt space. The space $\mathcal{DP}(X)$ is then rationally homotopy equivalent to $IX \cup e^{2s+1}$. If moreover $H^1(X_{reg}) = 0$, then e^{2s+1} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(IX) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by IX .*
- (2) *Suppose X is an L -space. The space $\mathcal{DP}(X)$ is then rationally homotopy equivalent to $\mathcal{I}^{\overline{\mathcal{L}}}X \cup e^{2s+1}$. If moreover $H^1(X_{reg}) = 0$, then e^{2s+1} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(\mathcal{I}^{\overline{\mathcal{L}}}X) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by $\mathcal{I}^{\overline{\mathcal{L}}}X$.*

Proof. The proof is the same as for the proposition 3.4.1 unless we consider the following diagram when X is a Witt space.

$$\begin{array}{ccccc} & L = \partial X_{reg} & \xhookrightarrow{i_1} & X_{reg} & \\ & \downarrow & & \downarrow & \\ S^{2s} & \xrightarrow{\phi} & tL & \xrightarrow{\quad} & IX \\ & \downarrow & \downarrow & & \downarrow \\ i_2 \downarrow & \text{(III) HPO} & t^\phi L & \xrightarrow{\quad} & \mathcal{DP}(X) \\ D^{2s+1} & \xrightarrow{\quad} & & & \end{array}$$

and the following diagram when X is an L-space.

$$\begin{array}{ccccc}
 L = \partial X_{reg} & \xrightarrow{i_1} & X_{reg} & & \\
 \downarrow & & \downarrow & \text{(I) HPO} & \\
 S^{2s} & \xrightarrow{\phi} & t^{\mathcal{L}} L & \xrightarrow{\quad} & \mathcal{I}^{\mathcal{L}} X \\
 \downarrow i_2 & \text{(III) HPO} & \downarrow & \text{(II)} & \downarrow \\
 D^{2s+1} & \xrightarrow{\quad} & t^{\mathcal{L}, \phi} L & \xrightarrow{\quad} & \mathcal{DP}(X)
 \end{array}$$

□

The considerations whether the approximations are good or very good come from the study of the rational homology of these diagrams of homotopy pushouts

$$\begin{array}{ccc}
 L \hookrightarrow X_{reg} & & L \hookrightarrow X_{reg} \\
 \downarrow & \searrow \phi & \downarrow \phi \\
 t^{\phi} L \longrightarrow \mathcal{DP}(X)_i & & t^{\mathcal{L}, \phi} L \longrightarrow \mathcal{DP}(X)_i \\
 \downarrow & \searrow \psi & \downarrow \psi \\
 * \longrightarrow X & & * \longrightarrow X
 \end{array}$$

for the even dimensional case and the Witt space case, or for the L-space case.

3.3. The multiple isolated singularities case. The theorem 3.2 did not make any assumptions on the connectivity of the pairs $(X_j, Y_j, [x_j])$, so in fact we can apply everything that was above to the case of a pseudomanifold with more than one isolated singularity.

Theorem 3.5 (Multiple isolated singularities case). *Let X be a compact, connected oriented normal pseudomanifold of dimension n with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$, $\nu > 1$, of links L_i simply connected. Then,*

- (1) *If $n = 2s$, there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X . Moreover, if $\dim X \equiv 0 \pmod{4}$, the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same as the Witt class associated to the middle intersection cohomology of X in $W(\mathbb{Q})$.*
- (2) *If $n = 2s + 1$ and X is either a Witt space or an L-space there exists a good rational Poincaré approximation $\mathcal{DP}(X)$ of X . Moreover is X is Witt space $\mathcal{DP}(X)$ is a very good rational Poincaré approximation of X*

Just like before, the considerations whether the approximations are good or very good come from the study of the rational homology of these diagrams of homotopy pushouts

$$\begin{array}{ccc}
 \bigsqcup_{\sigma_i} L_i \hookrightarrow X_{reg} & & \bigsqcup_{\sigma_i} L_i \hookrightarrow X_{reg} \\
 \downarrow & \searrow \phi & \downarrow \phi \\
 \bigsqcup_{\sigma_i} t^{\phi_i} L_i \longrightarrow \mathcal{DP}(X)_i & & \bigsqcup_{\sigma_i} t^{\mathcal{L}, \phi_i} L_i \longrightarrow \mathcal{DP}(X)_i \\
 \downarrow & \searrow \psi & \downarrow \psi \\
 \bigsqcup_{\sigma_i} * \longrightarrow X & & \bigsqcup_{\sigma} * \longrightarrow X
 \end{array}$$

for the even dimensional case and the Witt space case, or for the L-space case.

3.3.1. The even dimensional case. Let X be a compact oriented normal pseudomanifold of dimension $n = 2s$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of simply connected links L_i .

The rational Hurewicz theorem gives us maps ϕ_i such that the pairs $(t^{\phi_i} L_i, L_i)$ are Poincaré duality pairs for all i . Denote by $[e_\phi]_i$ the induced orientation class in $H_{2s}(t^{\phi_i} L_i, L_i)$.

The pair $(X_{reg}, \partial X_{reg})$ with $\partial X_{reg} = \sqcup_{\sigma_i} L_i$ is still a manifold with boundary, thus satisfies Poincaré-Lefschetz duality and is a Poincaré duality pair. The theorem 3.2 then applies and, with the same notation, $\mathcal{DP}(X)$ is an oriented Poincaré complex of dimension $2s$ without boundary and of orientation class given by

$$[\mathcal{DP}(X)] = i^{-1}(i_1 \oplus i_2([X_{reg}, \partial X_{reg}], [-e_\phi]_1, \dots, [-e_\phi]_r)),$$

and all the results obtained before remain true except for the proposition 3.4.1 which has to be modified. We have to take the normal intersection space $\mathcal{IX} = \mathcal{I}^m X = \mathcal{I}^n X$ to modify the proposition 3.4.1. Which becomes

Proposition 3.5.1. *Let X be a compact, connected oriented normal pseudomanifold of dimension $n = 2s$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected. Suppose moreover that $H^1(X_{reg}) = 0$. Then $\mathcal{DP}(X)$ is rationally homotopy equivalent to $t_{2s-1} \mathcal{IX} \cup e^{2s}$ where $t_{2s-1} \mathcal{IX}$ is the $(2s-1)$ -truncation of \mathcal{IX} and where e^{2s} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(t_{2s-1} \mathcal{IX}) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by $t_{2s-1} \mathcal{IX}$.*

Proof. Consider the following diagram, obtained by the construction of $\mathcal{DP}(X)$

$$\begin{array}{ccccc} & & \sqcup_{\sigma_i} L_i = \partial X_{reg} & \xhookrightarrow{i_1} & X_{reg} \\ & & \downarrow & & \downarrow \\ \sqcup_{\sigma_i} S_i^{2s-1} & \xrightarrow{\sqcup_{\sigma_i} \phi_i} & \sqcup_{\sigma_i} tL_i & \xrightarrow{\text{(I) HPO}} & \mathcal{IX} \\ & & \downarrow & & \downarrow \\ & & \sqcup_{\sigma_i} t^{\phi_i} L_i & \xrightarrow{\text{(II)}} & \mathcal{DP}(X) \\ \downarrow i_2 & \text{(III) HPO} & & & \\ \sqcup_{\sigma_i} D_i^{2s} & \longrightarrow & \sqcup_{\sigma_i} t^{\phi_i} L_i & \longrightarrow & \mathcal{DP}(X) \end{array}$$

With the same arguments than for the unique isolated singularity case, $\mathcal{DP}(X)$ is rationally homotopy equivalent to

$$\mathcal{DP}(X) \simeq \mathcal{IX} \cup \left(\bigcup_{\phi_i} e_i^{2s} \right).$$

Now, $H^1(X_{reg}) = 0$ so $\mathcal{DP}(X)$ is simply connected and the theorem 2.2 gives a rational homotopy equivalence

$$\varphi: \widetilde{\mathcal{DP}(X)} \longrightarrow \mathcal{DP}(X)$$

such that the differential in the integral cellular chain complex of $\widetilde{\mathcal{DP}(X)}$ is identically zero. This implies that there is only one top dimensional cell on $\widetilde{\mathcal{DP}(X)}$, we have a attaching map θ and a cell e^{2s} such that

$$\widetilde{\mathcal{DP}(X)} = X_0 \cup_{\theta} e^{2s}.$$

Let us now determine X_0 . By the theorem 2.2 X_0 is a CW-complex of dimension $2s-1$ and by the Poincaré duality of $\mathcal{DP}(X)$ we have

$$H_{2s-1}(X_0) = H_{2s-1}(\mathcal{DP}(X)) \cong H^1(\mathcal{DP}(X)) = H^1(X_{reg}) = 0.$$

For any other $r \leq 2s - 2$ we have by construction $H_r(X_0) = H_r(\mathcal{IX})$.

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{\text{incl}} & & & X_0 \cup_{\theta} e^{2s} \\
 \varphi| \downarrow & & & & \downarrow \varphi \\
 t_{2s-1}\mathcal{IX} & \xrightarrow{i} & \mathcal{IX}/2s-1 & \xrightarrow{h} & \mathcal{IX}^{2s-1} & \xrightarrow{j} & \mathcal{IX} \longrightarrow \mathcal{DP}(X)
 \end{array}$$

The above diagram, where the maps i and j are cellular inclusions and h a cellular homotopy equivalence as in the definition of the homology truncation 2.3.1, commutes and by the same argument that the one given before the proposition 2.3.2, φ restricts to a rational homotopy equivalence

$$\varphi|: X_0 \longrightarrow t_{2s-1}\mathcal{IX}.$$

Then, up to rational homotopy equivalence, we have

$$\mathcal{DP}(X) = t_{2s-1}\mathcal{IX} \cup_{\theta} e^{2s}.$$

The theorem 3.3 then tells us that e^{2s} is attached by ordinary Whitehead products (not iterated) with respect to some basis of $\pi_*(t_{2s-1}\mathcal{IX}) \otimes \mathbf{Q}$ and the rational homotopy type of $\mathcal{DP}(X)$ is determined by $t_{2s-1}\mathcal{IX}$. \square

In particular, if we have only one normal isolated singularity and if $\mathcal{DP}(X)$ is simply connected, then $H_{2s-1}(\mathcal{DP}(X)) = H_{2s-1}(\mathcal{IX}) = H_{2s-1}(\mathcal{IX}) = 0$ and $t_{2s-1}\mathcal{IX} \simeq \mathcal{IX}$. We then get back the proposition 3.4.1.

The others results remain true, in particular $\mathcal{DP}(X)$ is a rational Poincaré duality space and we have the first part of the theorem 1.2.

Proposition 3.5.2. *If $\dim X = 2s$, then $\mathcal{DP}(X)$ is a good rational Poincaré approximation of X . Moreover, if $\dim X \equiv 0 \pmod{4}$, then the Witt class associated to the intersection form $b_{\mathcal{DP}(X)}$ is the same that the Witt class associated to the middle intersection cohomology of X in $W(\mathbf{Q})$.*

3.3.2. *The odd dimensional case.*

Proposition 3.5.3. *Let X be a compact, connected oriented normal pseudomanifold of dimension $n = 2s + 1$ with only isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu; \nu > 1\}$ of links L_i simply connected.*

- (1) *Suppose moreover X is a Witt space, then $\mathcal{DP}(X)$ exists and is a very good rational Poincaré duality space,*
- (2) *Suppose moreover X is an L-space, then $\mathcal{DP}(X)$ is a good rational Poincaré approximation of X .*

Proof. The proof is the same as the ones for the propositions 3.4.2 and 3.4.5 but with multiple links L_i . \square

4. EXAMPLES AND APPLICATIONS

4.1. Real Algebraic varieties. If V is a real algebraic variety of even dimension with isolated singularities $\Sigma = \{\sigma_1, \dots, \sigma_\nu\}$ and an oriented regular part V_{reg} . We can apply the homological truncation and then by the use of the precedents results construct a rational Poincaré approximation $\mathcal{DP}(V)$.

The odd dimensional is more interesting. Suppose that V is a real algebraic variety of with multiple isolated singularities of odd dimension. If the regular part V_{reg} of V is oriented then V is automatically an L-space due to the following result of Selman Akbulut and Henry King :

Theorem 4.1 ([1]). *Let V be a compact topological space. Then the following are equivalent :*

- (1) *V is homeomorphic to a real algebraic set with isolated singularities.*
- (2) *V is homeomorphic to the quotient obtained by taking a smooth closed manifold M and collapsing each L_i to point a point where L_i , $i = 1, \dots, \nu$ is a collection of disjoint smooth subpolyhedra of M .*
- (3) *$V = M \cup \bigcup_{i=1}^{\nu} cL_i$ where M and L_i are smooth compact manifolds, ∂M is the disjoint union of the L_i 's, each L_i bounds a smooth compact manifolds and $L_i \times 1 \subset cL_i$ is identified with $L_i \subset M$.*

Consider then V a oriented real algebraic variety of dimension $n = 4s + 1$ with ν isolated singularities. Then by the third equivalence we have

$$V = M \cup \bigcup_{i=1}^{\nu} cL_i$$

and each link L_i is a smooth compact manifold of dimension $4s$ and is the boundary of a $4s + 1$ smooth compact manifold. Then by the lemma 3.4.3 we have that

$$[b_i] = 0 \in W(\mathbf{Q}) \forall i.$$

We can perform a Lagrangian truncation and we have our rational Poincaré approximation $\mathcal{DP}(V)$.

Note that if V is of dimension $4s + 3$ then we don't need this result because the bilinear form b_L would be skew-symmetric.

We have the following result

Proposition 4.1.1. *Every oriented real algebraic variety V with only isolated singularities and simply connected links admits at least a good rational Poincaré approximation $\mathcal{DP}(V)$.*

4.2. Hypersurfaces with nodal singularities. Let V be a complex projective hypersurface with one nodal singularity such that $\dim_{\mathbf{C}} V = 3$. The link of this singularity is $L = S^2 \times S^3$, by theorem 3.3 the link of the singularity is rationally homotopy equivalent to

$$L \simeq (S^2 \vee S^3) \bigcup_{\psi} e^5$$

where ψ is a Whitehead product.

Since $\overline{m}(6) = \overline{n}(6) = 2$, the homological truncation of the link is

$$t_2 L = S^2$$

and the cotruncation is rationally homotopy equivalent to

$$t^2 L = S^3 \vee S^5.$$

To see this, just compute the cohomology algebra of the cotruncation. The rational Hurewicz theorem 2.1 then says that we have the isomorphism

$$\pi_5(t^2 L) \otimes \mathbf{Q} \xrightarrow{\cong} H_5(t^2 L) \cong H_5(S^5).$$

The cell attachment ϕ obtained by this isomorphism then kill the 5-sphere of the cotruncation. That is we have

$$t^{\phi} L = (S^3 \vee S^5) \cup_{s_5^{\sharp}} e^6 \simeq S^3.$$

But $t^{\phi} L = S^3 \simeq D^3 \times S^3$ and since $\partial(D^3 \times S^3) = S^2 \times S^3$, the pair $(t^{\phi} L, L) = (D^3 \times S^3, S^2 \times S^3)$ is a Poincaré duality pair. The space $\mathcal{DP}(V)$ is a good rational Poincaré approximation of X .

This construction extends to multiple isolated singularities and higher dimension complex hypersurfaces with nodal singularities.

4.3. Thom Spaces.

Definition 4.1.1. Let B be a compact, connected, oriented manifold of dimension m and E a fiber bundle over B of rank m' ,

$$\mathbf{R}^{m'} \longrightarrow E \longrightarrow B.$$

The Thom space $Th(E)$ of the fiber bundle E is defined as the homotopy cofiber of the map

$$S_E \longrightarrow D_E$$

where S_E and D_E are respectively the sphere bundle and disk bundle associated to E .

$Th(E)$ is then a pseudomanifold of dimension $m + m'$, the singularity is the compactification point, its link is the sphere bundle S_E and the regular part of $Th(E)$ is the disk bundle D_E .

We show that in the case of an odd dimensional Thom space $Th(E)$ is either an L-space or a Witt space whether the rank of the vector bundle is lesser than the dimension of the base space or not.

Theorem 4.2. Suppose $m' > 0$.

- (1) Let $\mathbf{R}^{2m'} \longrightarrow E \longrightarrow B^{2m+1}$ with B be a manifold of dimension $2m+1$ and E a fiber bundle over B of rank $2m'$. Then,
 - if $m' \leq m+1$, $Th(E)$ is an L-space,
 - if $m' > m+1$, $Th(E)$ is a Witt space if and only if $H^{m+m'}(B) = 0$.
- (2) Let $\mathbf{R}^{2m'+1} \longrightarrow E \longrightarrow B^{2m}$ with B be a manifold of dimension $2m$ and E a fiber bundle over B of rank $2m'+1$. Then,
 - if $m' \leq m$, $Th(E)$ is an L-space,
 - if $m' > m$, $Th(E)$ is a Witt space if and only if $H^{m+m'}(B) = 0$.

Proof. Consider $\mathbf{R}^{2m'} \longrightarrow E \longrightarrow B^{2m+1}$.

In order to know if $Th(E)$ is an L-space or a Witt space we have to look at $H^{m+m'}(S_E)$ where S_E is the sphere bundle associated to the vector bundle E (see definitions 2.3.3 and 2.4.3). To compute $H^{m+m'}(S_E)$ we use the cohomological Leray-Serre spectral sequence associated to the fiber bundle

$$S^{2m'-1} \longrightarrow S_E \longrightarrow B^{2m+1}.$$

We have $E_2^{p,q} = H^p(B; H^q(S^{2m'-1})) = 0$ if $q \neq 0, 2m'-1$ and

$$d_{2m'} : E_{2m'}^{p, 2m'-1} \longrightarrow E_{2m'}^{p+2m', 0}$$

is the only non-zero differential which is defined by

$$d_{2m'} : E_{2m'}^{0, 2m'-1} \longrightarrow E_{2m'}^{2m', 0}$$

with $d_{2m'}(a) = eu(E) \in H^{2m'}(B)$ where a is the generator of $H^{2m'-1}(S^{2m'-1})$ and $eu(E) \in H^{2m'}(B)$ the Euler class of the sphere bundle. Since this is the only non-zero differential we have

$$H^{m+m'}(S_E) = E_{2m'+1}^{m+m', 0} \oplus E_{2m'+1}^{m-m'+1, 2m'-1}.$$

Suppose that $m' \leq m+1$.

The summand $E_{2m'+1}^{m-m'+1, 2m'-1}$ is well defined and using the structure product of the spectral sequence, we see that the product of two elements belonging to the same summand of $H^{m+m'}(S_E)$ is zero. Then by Poincaré duality the symmetric bilinear form

$$E_{2m'+1}^{m+m', 0} \times E_{2m'+1}^{m-m'+1, 2m'-1} \longrightarrow E_{2m'+1}^{2m+1, 2m'-1} \cong \mathbf{Q}\omega_a$$

induced by the product and where $\omega \in H^{2m+1}(B)$ is the fundamental class of the manifold B is non degenerate. Thus, provided than one of the two summand is non zero, the symmetric bilinear form is then hyperbolic and S_E is an L-space.

Suppose that $m' > m + 1$.

Then $E_{2m'+1}^{m-m'+1, 2m'-1} = 0$ and $E_{2m'+1}^{m+m', 0} = H^{m+m'}(B)$ and S_E is a Witt space if and only is $H^{m+m'}(B) = 0$.

We now consider $\mathbf{R}^{2m'+1} \rightarrow E \rightarrow B^{2m}$. By the same arguments we have

$$H^{m+m'}(S_E) = E_{2m'+2}^{m+m', 0} \oplus E_{2m'+2}^{m-m', 2m'}.$$

If $m' \leq m$ then $E_{2m'+2}^{m-m', 2m'}$ is well defined, the same arguments about the product structure and Poincaré duality imply that S_E is an L-space.

If $m' > m$ then $E_{2m'+2}^{m-m', 2m'} = 0$, $E_{2m'+2}^{m+m', 0} = H^{m+m'}(B)$ and $Th(E)$ is a Witt space if and only if $H^{m+m'}(B) = 0$. \square

Corollary 4.2.1. *For any complex line bundle $\mathbf{C} \rightarrow E \rightarrow B^k$ with $k \geq 2$, the Thom space $Th(E)$ is an L-space.*

Let $B^9 := (S^3 \times S^6) \sharp (S^4 \times S^5)$, where \sharp denotes the connected sum, and let $f : B^9 \rightarrow S^4$ the composition of the following contraction map q and projection map p :

$$B^9 \xrightarrow{q} S^4 \times S^5 \xrightarrow{p} S^4.$$

Let E be the fiber bundle over B^9 that is the pullback along f of the tangent space over S^4 ,

$$\begin{array}{ccc} \mathbf{R}^4 & & \mathbf{R}^4 \\ \downarrow & & \downarrow \\ E := f^*(TS^4) & \xrightarrow{\quad} & TS^4 \\ \downarrow & & \downarrow \\ B^9 & \xrightarrow{\quad f \quad} & S^4 \end{array}$$

The Thom space $Th(E)$ associated to this bundle is a pseudomanifold of dimension 13 and is an L-space by theorem 4.2.

Using the Leray-Serre spectral sequence we have

$$H^6(S_E) = \mathbf{Q}s_6 \oplus \mathbf{Q}s_3a.$$

where s_i represents the generator of the sphere S^i in B^9 and a the generator of the fiber S^3 of the sphere bundle. Denote by ω the fundamental class of B^9 with $\omega = s_4s_5 = s_3s_6$.

Using the product structures of the spheres S^6 and S^3 , we have $s_6s_6 = 0$ and $(s_3a)(s_3a) = 0$, but since $\omega = s_3s_6$ the matrix of the intersection form

$$H^6(S_E) \times H^6(S_E) \rightarrow \mathbf{Q}$$

in the base (s_6, s_3a) is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The intersection form is then hyperbolic and both factors $\mathbf{Q}s_6$ and $\mathbf{Q}s_3a$ are Lagrangian subspaces.

We now construct a rational model of $\mathcal{DP}(Th(E))$. For that we'll need a surjective model of $S_E \hookrightarrow D_E$ and a model of the Lagrangian truncation $t^{\overline{\mathcal{L}}}S_E$. A surjective model of $S_E \hookrightarrow D_E$ is given by

$$A(D_E) \xrightarrow{\varphi} A(S_E)$$

with

$$\begin{cases} A(S_E) &= (A(B) \otimes \wedge a, d) \text{ with } da = s_4 \\ A(D_E) &= (A(B) \otimes \wedge(a, b), D) \text{ with } Da = s_4 - b \\ \varphi|_{A(B) \otimes \wedge a} &= \text{id} \\ \varphi(b) &= 0 \end{cases}$$

where $A(B)$ is a rational model of the base space $(S^3 \times S^6) \sharp (S^4 \times S^5)$ which is given by

$$A(B) = (\wedge(s_3, s_4, s_5, s_6, \beta_6, \beta_{7,1}, \beta_{7,2}, \beta_8, \dots), d)$$

with $|s_i| = |\beta_i| = i$ and

$$\begin{cases} d\beta_6 &= s_3 s_4 \\ d\beta_{7,1} &= s_4^2 \\ d\beta_{7,2} &= s_5 s_3 \\ d\beta_8 &= s_3 s_6 - s_4 s_5. \end{cases}$$

Since $\dim B = 9$ we only gave elements of the model of B up to degree 9, the rest of the model being an acyclic part. That is for every element α_k of degree $k \geq 10$ such that $d\alpha_k = 0$, there is an element β_{k-1} such that $d\beta_{k-1} = \alpha_k$. In fact we can take a better model for $A(S_E)$ and $A(D_E)$ because the base space B is a formal space, that is we have a quasi isomorphism

$$\psi : (A(B), d) \longrightarrow (H(B), 0)$$

given by

$$\begin{cases} \psi(s_i) &= s_i \\ \psi(\beta_i) &= 0 \\ \psi(A^{\geq 10}(B)) &= 0. \end{cases}$$

The models we use are then

$$\begin{cases} A(S_E) &= (H(B) \otimes \wedge a, d) \text{ with } da = s_4 \\ A(D_E) &= (H(B) \otimes \wedge(a, b), D) \text{ with } Da = s_4 - b \\ \varphi|_{H(B) \otimes \wedge a} &= \text{id} \\ \varphi(b) &= 0 \end{cases}$$

By adapting the proposition 2.3.2 and then using the lemma 2.3.1 we can show that a model of the Lagrangian cotruncation is given by

$$(A(t^{\overline{\mathcal{L}}}S_E), d) = (\mathbf{Q} \oplus I_{\mathcal{L}}, d)$$

where $I_{\mathcal{L}}$ is the differential ideal given in this case by a choice of generator for one of the Lagrangian subspaces, a complementary of $\ker(d : A^6(S_E) \rightarrow A^7(S_E))$ and all the cochains of degree greater or equal to 7 of $A(S_E)$. A choice of $A(t^{\overline{\mathcal{L}}}S_E)$ is given by

$$\mathbf{Q} \oplus (\mathbf{Q}_{s_6} \otimes (H(B) \otimes \wedge a)^{\geq 7})$$

and the model of $I^{\overline{\mathcal{L}}}Th(E)$ is

$$A^*(I^{\overline{\mathcal{L}}}Th(E)) = (H(B) \otimes \wedge(a, b), D) \oplus_{H(B) \otimes \wedge a} (\mathbf{Q} \oplus I_{\mathcal{L}}, d).$$

We attach the top cell inducing Poincaré duality by Whitehead products with respect to some basis of $\pi_*(I^{\mathbb{C}}Th(E)) \otimes \mathbf{Q}$ and denote the resulting space by $\mathcal{DP}(Th(E))$, its cohomology algebra is then

$$H^*(\mathcal{DP}(Th(E))) = \frac{\mathbf{Q}[e_4, e_6] \otimes \wedge(e_7, e_9, e_{13})}{(e_4^2, e_6^2, e_4e_6, e_4e_7, e_6e_9, e_7e_9, e_6e_7 = e_4e_9 = e_{13})}, |e_i| = i.$$

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